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ALMOST PERIODIC POINTS AND MINIMAL SETS IN FORT' SPACES

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ABSTRACT
The given paper provides a brief introduction to dynamical systems and some light on the idea of Fort space and several known results concerning almost periodic points and minimal sets of maps are discussed.
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KEYWORDS: Separation axioms; Fort space; Recurrent point; Fixed points, Orbits; Periodic point; Almost periodic point; Minimal set.

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1. INTRODUCTION

All the number system such as sets of real numbers, integers, non-negative integers, positive integers and natural numbers mainly denoted by P, Z, Z_0, Z_+ and N respectively play an important role in dynamical system.

Dynamical system is a collection of probable states governed by certain rules to determine the current phase of the system in context of the past state.

Dynamical system is categorized mainly into two different forms. When the dynamical system is restricted by discrete time, it is termed as discrete-time system/dynamical system. In such system, input and output are the current state and new state respectively. State of the system is a rule which is governed by some information referred as input for the system [**1**]. Composition of mapping is an evolutionary process of studying dynamical system[5]**.** If $f:S\to S, S\neq \phi$, then by using mathematical induction, you can obtained the functions $f^1, f^2, f^3, \ldots, f^{n-1}, f^n\left(=f^{n-1}of\right)$ using iteration [**2**].

Dynamical systems have different notations for its categorised forms. Its discrete form is defined as $f:X\to X, X\neq\phi$, where $f^{n+1}=f^n\,o\,f, \forall\, n\in N$ and $f^0=I\left(\text{Identity function}\right)$. In dynamical system the composite mapping is defined as $f^{m+n}=f^m\,o\,f^n, \forall m,n\in ¥$ and $f^n=f\,f^{n-1}\,\forall\, n\in ¥$. The invertible mapping for isomorphic function *f* is defined as $f^{-n}=\left(f^n\right)^{-1}, \forall n\in ¥$. For the dynamical systems $\left(X,f\right)$ and $\left(Y,g\right)$ we get a new dynamical system $h\left(x,y\right)=\left(f\left(x\right),g\left(y\right)\right); x\in X, y\in Y$ and its inverse function is defined as $h^{-1}\left(x,y\right)=\left(f^{-1}\left(x\right),g^{-1}\left(y\right)\right); x\in X, y\in Y$.

The second main category of dynamical system is the limiting case of discrete dynamical systems where inputs are updated even for very small interval of time. For such system, the governing rule consists of sequence of differential equations; which referred as continuous-time dynamical system.

In dynamical system forecasting of developing system is performed with the passage of time. Thus, for given value of *x*, you will get the function $f^k\left(x\right)$ for large *k*. In dynamical system $\left(X,f\right)$, for any point $a\in X$, the set of points $\left\{a, f\left(a\right), f^2\left(a\right), f^3\left(a\right), \ldots, f^{n-1}\left(a\right), f^n\left(a\right)\right\}$ denoted by $O\left(a,f\right)$ termed as orbit of '*a*' and '*a*' is known as the seed or initial value of the orbit **[1].** Here, function *f* is termed as map, since, its domain and range both are the same.

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A point $a \in X$ is called a recurrent point of *f* and $O(a, f)$ is called a *recurrent orbit* if for any neighbourhood N_a of '*a*' any $m \in N$ ∃ an integer $n > m$ such that $f^n(a) \in N_a$.

Some time the functional value at any stage becomes equal to its initial value. The initial value *a* is termed as fixed point of the map, i.e., $f(a) = a$ [**2**]. The iterated value of the function $p: X \to X, X \neq \phi$ defined by $p(x) = 2x(1-x)$ are given by {0.01, 0.0198, 0.0388,..} represents the orbits of *p*. Here, $p(a) = a$ for $a = 0, \frac{1}{2}$, thus these are the fixed point of *p*. It is not necessary that every point of an orbit is a fixed point; several points on the map of the function which are not fixed, termed as **eventually fixed**. For example, $x = -1$ which is a seed of the function *f* defined by $f(x) = x^2$ is eventually fixed since $f(-1) = 1 \neq -1$ but $f^2(-1) = f(f(-1)) = f(1) = 1$. Thus, the point 1 lying on the orbit of *x* is fixed which shows that *x* is eventually fixed [**2**]. The fixed point of the orbit may be stable or unstable that can be checked using differentiation. The analytic real functions which are always continuous recall as smooth functions.
If $f: R \to R$ and *a* is any its fixed point. Then,

- *a* is of attracting nature if ∃ $I = (\alpha, \beta)$ and $a \in I$, if $x \in I$ then $\lim_{n\to\infty} f^n(x) = a$
- *a* is of repelling nature if ∃ $I = (\alpha, \beta)$ and $a \in I$, if $x \in I$ then $\lim_{n\to\infty} f^n(x) \neq a$
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There are some points termed as neutral fixed point which is neither attracting nor repelling [**3**].

There is some point on a map whose value is same as the functional value of some iterated function; this value is termed as periodic point. Thus, point α is known as a periodic point of period *m* or period-*m* point if for any least positive integer *m* such that $f^m(\alpha) = \alpha$ and the orbit associated with this point is recall as period-*m* orbit.

For example, the function *f* defined by $f(x) = x^2 - 1$ has a periodic point α = 0 of period *m* = 2, since $f^2(\alpha) = f(f(\alpha)) = f(-1) = 0 = \alpha$.

It is not necessary that every seed α of the orbit becomes periodic point, there are some points in the orbit of α which becomes periodic point and then α will be eventually periodic point [**2**].

The dynamical system is governed by the time factor[**3**]**.** Thus, the metric *d* and continuous function *f* defined over the non-empty set *X* constitute a dynamical system (X, f) [13]. The given system is said to be trivial if it contain single point, for such system mapping is said to be an identity mapping which is always unique[**3**]. If the metric *d* defined over a compact metric space *X* having a countable basis the given dynamical system (X, f) recall as *topological dynamical system*.

A discrete dynamical system becomes a topological dynamical system if the continuous mapping *f* is one-one, onto and f^{-1} is continuous. For this case topological dynamical system is also invertible. If there is homeomorphism *h* between the metric spaces (X_1, d_1) and (X_2, d_2) respectively such that $f_2 \, o \, h = h \, o \, f_1$ i.e., $f_2(h(x)) = h(f_1(x))$ $\forall \, x \in X_1$, then the dynamical systems (X_1, f_1) and (X_2, f_2) defined over the given metric spaces are said to be *conjugate dynamical systems*.

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If for any seed a of the system X, if its orbit $\{a, f(a), f^2(a), f^3(a), \ldots, f^{n-1}(a), f^n(a), \ldots\}$ is dense in X, then the dynamical system (X, f) is said to be *transitive*[**13**]. Also f is isometric if it preserves the distance[**1**].

For each member x of (X, f) its orbits is the set denoted by $\text{Orb}(x, f) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ and its limiting case is $\omega(x, f) = \bigcap_{n=1}^{\infty} \overline{\{f^i(x), i \geq n\}}$

Any point x of (X, f) is called an *almost periodic point* of f and $O(x, f)$ is called an *almost periodic orbit*

if for any neighbourhood N_x of x $\exists\, k \in N$ such that $f^{n+i}(x) \cap N_x \neq \phi;\ i = 1,\ 2,\ 3, \ldots k\ \forall n \in Z_+$.

Every component of limiting case is recall as recurrent point[**3**] and its collection $R(f)$ is f-invariant. Periodic points are always recurrent. In (X, f) if f is invertible, then α *-limit set* of x is $\alpha(x) = \alpha(x, f) = \bigcap_{n \in N} \overline{\bigcup \{f^{-i}x, i \geq n\}}$. Both the sets $\omega(x, f)$ and $\alpha(x, f)$ are closed and f-invariant. Point $x \in X$ (phase space) is *non-wandering* point if for any neighbourhood G of x $\exists\, n \in ¥$ such that $f^n(G) \cap G = \phi$. The set of non-wandering points $N_W(x, f)$ is closed, f -invariant, and contains the points of $\omega(x, f)$ and $\alpha(x, f)$ for all the members x of $X \Rightarrow R(f) \subset N_W(x, f)$. If Y is a non-empty closed invariant subset of X, then (Y, f) is also a dynamical system recall as a subsystem of (X, f).

If $\phi \neq Y \subset X$ where $\overline{Y} = Y$ & $f(Y) \subset Y$, then Y is f-minimal subset of X. A compact invariant set Y is minimal iff $f^{n+i}(y) \subset Y\ \forall\, y \in Y, i \in ¢_+$ and $f^{n+i}(y)$ is dense in Y. Thus, a periodic orbit is a minimal set. Topological dynamical system is consistently minimal if and only if it does not have any proper subsystem. Thus, any non-empty closed invariant subset Y of X is minimal iff $f^n(y_i) \cap \mathrm{Y} \neq \phi\ \forall\, y_i \in Y, n \in ¥$. The member x of the phase space X termed as minimal point or almost periodic point if it is the member of minimal set.

In this paper we raise the concept of Fort space. For this we will generalize the results of Gottschalk [6] to Fort spaces. Our main aim is, by means of such a generalization, to find some more essential connections between almost periodic points and minimal sets in Fort's space.

2. FORT TOPOLOGY AND FORT'S SPACE

Theorem 2.1 Let X be any uncountable set and let α be a fixed point of X. Let $\tau = \{G | G \subseteq X, \alpha \notin G\} \cup \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{ is finite}\}$. Then τ is a topology on X.

Proof:

Let us consider two space τ_1 and τ_2 such that $\tau_1 = \{G | G \subseteq X, \alpha \notin G\}$ and $\tau_2 = \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{ is finite}\}$, then $\tau = \tau_1 \cup \tau_2$

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(I) $\alpha \notin \phi \Rightarrow \phi \in \tau, \alpha \in X$ and $X - X = \phi$ is a finite set $\Rightarrow X \in \tau_2 \Rightarrow X \in \tau$

(II) Let $A_1, A_2 \in \tau$.

Case I: Let $A_1,\ A_2 \in \tau_1$. Then, $\alpha \notin A_1$ and $\alpha \notin A_2$. Hence $\alpha \notin A_1 \cap A_2$.

Thus, $A_1 \cap A_2 \in \tau_1 \Rightarrow A_1 \cap A_2 \in \tau$

Case II: Let $A_1,\ A_2 \in \tau_2$. If $A_1 \in \tau_2 \Rightarrow \alpha \in A_1$ and $X - A_1$ is finite.

If $A_2 \in \tau_2 \Rightarrow \alpha \in A_2$ and $X - A_2$ is finite.

$\Rightarrow \alpha \in A_1 \cap A_2$ and $X - (A_1 \cap A_2) = (X - A_1) \cup (X - A_2)$ is finite.

Hence, $A_1 \cap A_2 \in \tau_2 \Rightarrow A_1 \cap A_2 \in \tau.$

Case III: Let $A_1 \in \tau_1$ and $A_2 \in \tau_2$.

Then, $\alpha \notin A_1 \Rightarrow \alpha \notin A_1 \cap A_2 \Rightarrow A_1 \cap A_2 \in \tau_1 \Rightarrow A_1 \cap A_2 \in \tau.$

Case IV: Let $A_2 \in \tau_1$ and $A_1 \in \tau_2$.

Then, $\alpha \notin A_2 \Rightarrow \alpha \notin A_1 \cap A_2 \Rightarrow A_1 \cap A_2 \in \tau_1 \Rightarrow A_1 \cap A_2 \in \tau.$

It is concluded that in all the four cases, the membership of set A_1 and A_2 in the space τ shows that their common members are the members of the space τ, i.e., $A_1 \cap A_2 \in \tau.$

(iii) Let $A_\lambda \in \tau\ \forall \lambda \in \wedge$ (indexed set).

If $A_\lambda \in \tau_1\ \forall \lambda \in \wedge$, then $\alpha \notin A_\lambda\ \forall \lambda \in \wedge \Rightarrow \bigcup_{\lambda \in \wedge} A_\lambda \in \tau_1.$

Hence, $\bigcup_{\lambda \in \wedge} A_\lambda \in \tau.$

Now for $\lambda_0 \in \wedge$ such that $A_{\lambda_0} \notin \tau_1$ this infers that $A_{\lambda_0} \notin \tau_2$.

This shows $\alpha \in A_{\lambda_0}$ and $X - A_{\lambda_0}$ is a finite set.

Now, $A_{\lambda_0} \subseteq \bigcup_{\lambda \in \wedge} A_\lambda \Rightarrow \alpha \in A_{\lambda_0} \subseteq \bigcup_{\lambda \in \wedge} A_\lambda$ and $X - \bigcup_{\lambda \in \wedge} A_\lambda \subseteq X - A_\lambda \bigcup_{\lambda \in \wedge} A_\lambda$.

As $X - A_{\lambda_0}$ is a finite set, then, $X - \bigcup_{\lambda \in \wedge} A_\lambda$ will also become finite set.

Therefore, in this situation $\bigcup_{\lambda \in \wedge} A_\lambda \in \tau_2 \Rightarrow \bigcup_{\lambda \in \wedge} A_\lambda \in \tau.$

Hence, after observation we can say that in either case $A_\lambda \in \tau\ \forall \lambda \in \wedge \Rightarrow \bigcup_{\lambda \in \wedge} A_\lambda \in \tau.$

After the study of (i), (ii) and (iii) we can say that τ is a topology on *X*.

This topology τ is called **Fort's topology** on *X* and the topological, i.e. T-space (X, τ) is called **Fort's space**.

Separation Axioms

T_0- space: A topological space (X, τ) is said to be a T_0 – space or Kolomogrov space, if x_1 and x_2 are any two distinct points of (X, τ) then $\exists$ an open set $G \in \tau$ such that if $x \in G$ then $y \notin G$ and vice versa.

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T_1- space: A topological space (X,τ) will be a T_1 – space or **Fréchet space**, if for $x_1, x_2 \in (X,\tau); x_1 \neq x_2$ then there exists open sets $G_{x_1}, G_{x_2} \in \tau$ such that if $x \in G_{x_1}, G_{x_2} \in \tau$ then $y \notin G_1$ and if $y \in G_2$ then $x \notin G_2$.

T_2- space: A topological space (X,τ) is said to be a T_2 – space or Hausdorff space, if x_1 and x_2 are any two distinct points of (X,τ) then there exists open sets $G_{x_1}, G_{x_2} \in \tau$ such that if $x_1 \in G_{x_1}$ and $x_2 \in G_{x_2}$ then $G_{x_1} \cap \in G_{x_2} = \phi$.

T_3- space: A topological space (X,τ) is said to be a T_3 – space, if A is a closed set and b is not the member of A then there exists disjoint open sets $G_a, G_b \in \tau$ such that $b \notin A \subseteq G_a \cap G_b = \phi$.

T_4- space: A topological space (X,τ) is said to be a T_4– space, if A and B are disjoint closed sets in (X,τ) then there exists disjoint open sets $G_A, G_B \in \tau$ such that $A \subseteq G_A$ & $B \subseteq G_B$. In other words (X,τ) is said to be a T_4– space, if and only if every open set G contains a closed neighborhood of each closed set contained in G.

T_5- space: A topological space (X,τ) is said to be a T_5– space, if A and B are separated sets in (X,τ) then there exists open sets $G_A, G_B \in \tau$ such that $A \subseteq G_A$ & $B \subseteq G_B$.

Regular space: A topological space (X,τ) is said to be *regular* if for any closed set $F \subset X$ and any point $x \in X - F$ there exists disjoint open sets $G_1, G_2 \in \tau$ such that $x \in G_1$ and $F \in G_2$.

In other way a topological space (X,τ) is said to be regular if it satisfies the following axiom. "If F is a closed subset of X and x is a point of X not in F, Then there exist a continuous function $f : X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

A regular T_1-space is called a T_3-space. We now raise a notion of ω-regular space.

***ω*-regular space:** Since, a Fort space is defined by $\tau = \{G | G \subseteq X, \alpha \notin G\} \cup \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{ is finite}\}$. Now the Fort space (X,τ) is said to be an *ω-regular space* if for any closed set $F \subset X$ and any point $x \in X - F$ and any countable set $A \subset W$, there exists disjoint open sets $G_1, G_2 \in \tau$ such that $x \in G_1$ and $A \subseteq G_2$.

It follows from the definition that all regular spaces are ω-regular. But converse need not be true, i.e., ω-regular space not necessary a regular space. Thus the notion of ω-regular space is more extensive than that of regular space.

Closure and Compactness of Fort's space

Theorem 2.2 Every Fort space is compact space.

Proof. Let X be an uncountable set and let α be a fixed point of X.

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Let $\tau = \{G | G \subseteq X, \alpha \notin G\} \cup \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{ is finite}\}$ .Then (X, τ) is a topological space and thereby a Fort space.
In order to prove that (X, τ) is compact space, prove that every open cover of *X* has a sub-cover.

Let $G = \{G_\eta | \eta \in \Lambda\}$ be any open cover for *X*, i.e., $X = \bigcup_{\eta \in \wedge} G_\eta$.

Since, $X = \bigcup_{\eta \in \wedge} G_\eta$ and $\alpha \in X$, we obtained $\alpha \in G_{\eta_0}$ for some $\eta_0 \in \Lambda$.

By the definition of (X, τ), $G_{\eta_0} \in \tau \Rightarrow X - G_{\eta_0}$ is a finite subset of *X*.

Let $X - G_{\eta_0} = \{x_1, x_2, \ldots, x_n\}$. Choose $G_{\eta_i} \in \{G_\eta | \eta \in \Lambda\}$ such that $x_i \in G_{\eta_i} \forall i, i \in [1, n]$.

Therefore, $X = G_{\eta_0} \cup (X - G_{\eta_0}) \subseteq G_{\eta_0} \cup G_{\eta_1} \cup \ldots \cup G_{\eta_n}$

And thereby $X = \bigcup_{\eta \in \wedge} G_\eta = G_{\eta_0} \cup G_{\eta_1} \cup \ldots \cup G_{\eta_n}$.

This infers that every open cover $G = \{G_\eta | \eta \in \Lambda\}$ of *X* possess finite sub-cover and this shows that every Fort space is compact, i.e., (X, τ) is compact.

Theorem 2.3 Every Fort space is countably compact.

Proof:
As here it is proved that every Fort space is compact space, thus, it requires to prove that every compact space is countable compact.
Consider that there exists a Fort space (X, τ), which is not countably compact. Hence, there exists an infinite set $A \subseteq X$ having no limit point in *X*. Thus, each $x \in X$ is not a limit point of A. But then for each $x \in X$, $\exists$ an open set G_x contains *x* such that

$G_x \cap A - \{x\} = \phi$.

$\Rightarrow$ Either $G_x \cap A = \phi$ or $G_x \cap A = \{x\} \forall x \in X$.

Again as $\{G_x\}_{x \in X}$ forms an open cover for *X*, which is already a compact space, hence *X* have finite sub-cover.

Let $X = \bigcup_{i=1}^{n} G_{x_i} \Rightarrow A = X \cap A = \left[\bigcup_{i=1}^{n} G_{x_i}\right] \cap A = \bigcup_{i=1}^{n} [G_{x_i} \cap A]$

Since, $G_{x_i} \cap A = \{x_i\}$ or $G_{x_i} \cap A = \phi \ \forall i, i \in [1, n]$

$\Rightarrow \bigcup_{i=1}^{n} [G_{x_i} \cap A]$ must be a finite set.

Hence A is a finite set of X, which is a contradiction.

Thus, or supposition is wrong, hence every compact space is countably compact and thereby the Fort space are countably compact space.

Theorem 2.4 Every Fort space is a T_2 – space, i.e., a Hausdorff space.

Proof:

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Let X be an uncountable set and let α be a fixed point of X.

Let $\tau = \{G | G \subseteq X, \alpha \notin G\} \cup \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{is finite}\}$ .Then (X,τ) is a topological space and thereby a Fort space.

Let us consider two space τ_1 and τ_2 such that $\tau_1 = \{G | G \subseteq X, \alpha \notin G\}$ and $\tau_2 = \{G | G \subseteq X, \alpha \in G \text{ and } X - G \text{ is finite}\}$, then $\tau = \tau_1 \cup \tau_2$ is a topology on X.

We need to prove that (X,τ) is a T_2 – space, for this consider two distinct points x and y of X.

Case I: Let x and y both are different from the fixed point α.

Let $G = \{x\}$ and $H = \{y\}$, then $G \cap H = \phi$

Case II: Let $y = \alpha$. Then $G = X - \{\alpha\}$ and $H = \{\alpha\}$ are disjoint open sets containing x and α respectively.

Thus from both the cases we conclude that (X,τ), i.e., the Fort's space is a Hausdorff space.

A topological space (X,τ) is said to be regular if it satisfies the following axiom.

"If $F(closed) \subseteq (X,\tau)$ and $x \in X \,\&\, x \notin F$, then $\exists$ a continuous function $f : X \to [0,1]$ such that $f : X \to [0,1]$ $f(x) = 0$ and $f(F) = \{1\}$.

Definition: Family of sets holds finite intersection property if its each finite subset have non-void intersection.

Theorem 2.5 A topological space (X,τ) is compact if and only if every family of closed sets having the finite intersection property has a non-empty intersection.

Proof: Necessary Condition:

Let (X,τ) be compact space and let $F = \{F_\eta | \eta \in \Lambda\}$ be a family of closed sets in X holds finite intersection property.

To prove that $\bigcap_{\eta\in\Lambda} F_\eta \neq \phi$.

Suppose $\bigcap_{\eta\in\Lambda} F_\eta = \phi$, then $\left(X - \bigcap_{\eta\in\Lambda} F_\eta\right) = X \Rightarrow \bigcup_{\eta\in\Lambda}(X - F_\eta) = X$.

Thus the family $\{X - F_\eta | \eta \in \Lambda\}$ forms an open cover for (X,τ) As (X,τ) is compact, this open cover has finite sub-cover.

Let $X = \bigcup_{i=1}^{n}(X - F_{\eta_i})$. But then $\bigcap_{i=1}^{n} F_{\eta_i} = \phi$ a contradiction to our assumption.

Hence $\bigcap_{\eta\in\Lambda} F_\eta \neq \phi$

Sufficient Condition:

Let any family of closed sets in (X,τ) holds finite intersection property.

To prove that (X,τ) is compact. Let suppose (X,τ) is not compact. Then there exist an open cover ofC such that $X \neq \bigcup_{i=1}^{n} G_{\eta_i}$ for any finite n.

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Hence $\bigcap_{i=1}^{n}\left(X-G_{\eta_i}\right)\neq\phi$ for any finite n.

Thus the family $\left\{X-G_{\eta}\middle|\eta\in\Lambda\right\}$ of closed sets holds finite intersection property. Hence by assumption $\bigcap_{i=1}^{n}\left(X-G_{\eta_i}\right)\neq\phi$, i.e., $X\neq\bigcup_{\eta\in\Lambda}G_{\eta}$ which is a contradiction.

Hence our assumption is wrong. Therefore (X,τ) must be compact space.

Theorem 2.6 A T – space (X,τ) is compact if and only if every basic open cover of has a finite subcover.

Proof: Let (X,τ) be a compact. Then every open cover of (X,τ) has a finite subcover. In general, every basic open cover of (X,τ) must have a finite subcover.

Conversely, suppose that every basic open cover of (X,τ) has a finite subcover and let

$C=\left\{G_{\eta}\in\tau\middle|\eta\in\Lambda\right\}$ be any open cover of (X,τ). If $\mathsf{B}=\left\{B_{\alpha}\middle|\alpha\in\Lambda\right\}$ be any open base for (X,τ), then each G_{η} is union of some members ofB and the totality of all such

members of B is evidently a basic open cover of (X,τ). By hypothesis this collection of members of B has a finite subcover, say $\left\{B_{\alpha_i}\middle|i=1,2,...,n\right\}$.

For each B_{α_i} in this finite subcover, we can select a G_{η}, from X such that $B_{\alpha_i}\subset G_{\eta_i}$.
It follows that the finite subcollection
$\left\{G_{\eta_i}\middle|i=1,2,...,n\right\}$ Which arises in this way is a subcover of X.

Hence (X,τ) is compact.

Theorem 2.7 Every closed subset of a compact space is compact i.e. compactness is closed Hereditary.

Proof: Let (X,τ) be a compact space and *F* be a closed subset of (X,τ).

To prove that *F* is compact. Let $C=\left\{G_{\eta}\in\tau\middle|\eta\in\Lambda\right\}$ be any open cover of *F*.

As $X=F\cup(X-F)\subseteq\left[\bigcup\left\{G_{\eta}\in\tau\middle|\eta\in\Lambda\right\}\right]\bigcup(X-F)$ shows that forms an open cover for (X,τ).

As (X,τ) is compact, this open cover has a finite sub-cover.

Let $X=\bigcup_{i=1}^{n}\left\{G_{\eta_i}\middle|\eta_i\in\Lambda\right\}\bigcup(X-F)$. Then surely $F\subseteq\bigcup_{i=1}^{n}G_{\eta_i}$

Thus the open cover $C=\left\{G_{\eta}\in\tau\middle|\eta\in\Lambda\right\}$ of *F* have predetermined sub-cover. Hence *F* is compact.

Theorem 2.8 An intersection of closed compact sets of a T-space is a closed compact set.

Proof: Let (X,τ) be compact space and let $\left\{F_{\eta}\in\tau\middle|\eta\in\Lambda\right\}$ be a family of closed compact sets in (X,τ).

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To prove that $\bigcap_{\eta\in\Lambda} F_\eta$ is a closed compact set in (X,τ)

Obviously, $\bigcap_{\eta\in\Lambda} F_\eta$ is a closed set in (X,τ) as F_η is a closed set for each $\eta\in\Lambda$

Now $\bigcap_{\eta\in\Lambda} F_\eta \subseteq F_\eta$ shows that $\bigcap_{\eta\in\Lambda} F_\eta$ is a closed subset of a compact set F_η.

Hence, $\bigcap_{\eta\in\Lambda} F_\eta$ _is a compact set

Theorem 2.9 Let (X,τ) be a topological space. Let $\tau^*\le\tau$. Then (X,τ^*) is a compact space.

Proof: Let $C=\{G_\eta\in\tau \mid \eta\in\Lambda\}$ be an τ^*-open cover for S. Since, $\tau^*\le\tau$, therefore $C=\{G_\eta\in\tau \mid \eta\in\Lambda\}$ is also τ-open cover for (X,τ).

As (X,τ) is compact, there exist a finite sub-cover say $C^*=\{G_{\eta_i}\in\tau^* \mid \eta_i\in\Lambda, 0\le\eta_i\le\pi\}$ for (X,τ).

But this in turns shows that (X,τ^*) is compact.

Theorem 2.10 Consider two topological spaces (X,τ) and (Y,τ^*) where (X,τ) is a compact space and let $f:X\to Y$ be onto, continuous map. Then (Y,τ^*) should be of same category.

Proof: Suppose $C=\{G_\eta^*\in\tau^* \mid \eta\in\Lambda\}$ is an open cover for (Y,τ^*).

As $Y\subseteq\bigcup_{\eta\in\Lambda} G_\eta^*$ and *f* is onto and continuous map we get $X=f^{-1}\left[\bigcup_{\eta\in\Lambda} G_\eta^*\right]=\bigcup_{\eta\in\Lambda} f^{-1}(G_\eta^*)$,

$G_\eta^*\in\tau^*, \forall\eta\in\Lambda$ and we get $f^{-1}(G_\eta^*)\in\tau, \forall\eta\in\Lambda$

Hence $\{f^{-1}(G_\eta^*)\in\tau \mid \eta\in\Lambda\}$ forms an open cover for (X,τ). As (X,τ) is compact this open cover has a finite sub-cover. Let $X=\bigcup_{i=1}^{n} f^{-1}(G_{\eta_i}^*)$. But $Y=\bigcup_{i=1}^{n} f^{-1}(G_{\eta_i}^*)$;

$\Rightarrow C=\{G_\eta^*\in\tau^* \mid \eta\in\Lambda\}(\text{an open covering of } Y)$ have finite sub-cover.

Hence, (Y,τ^*) is compact.

Corollary 2.11 Every compact space is a topological space.

Corollary 2.12 Let $f:(X,\tau)\xrightarrow[\text{onto}]{\text{continuous}}(Y,\tau^*)$ associates every compact element of *X* onto a compact subset of *Y*.

Proof: Let E is a compact subset of (X,τ). Restriction of *f* on the subspace E of (X,τ) is continuous onto map on the subspace *f*(E) of *Y*. Hence, $f(E)$ is a compact space of *Y*.

One point compactification

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We know that every topological space need not be compact. But for non-compact space (X,τ) we can formed a compact space (X^*,τ^*) such that X is homeomorphic with some dense subspace of X. This compact space (X^*,τ^*) is called compactification of the space X. If $X^* = X \cup \{\infty\}$, for some object $\infty \notin X$ then compactification of X is called one-point compactification.

A topological space is (X,τ) is a locally compact space if each point $x \in X$ has a compact neighbourhood.

Theorem 2.13 Every compact space is locally compact.
Proof: Let (X,τ) be a compact space. Then for any $x \in X$, X itself is a compact neighbourhood of x. Hence X is locally compact.

Theorem 2.14 Closed subset of a locally compact space is locally compact.
Proof:- Let (X,τ) be a locally compact space and let F be any closed subset of X . For showing the subspace (F,τ^*) is locally compact consider x be the member of closed set F. As $x \in X$ and X is locally compact, then there exist a compact neighbourhood say K of x in (X,τ). As $F \cap K$ is a closed subset of a compact space K, we get $F \cap K$ is compact neighbourhood of x in F. Hence (F,τ^*) is locally compact.

Countably compact spaces
A topological space (X,τ) is said to be countably compact if any infinite subset of X has a limit point.

Theorem 2.15 Every compact space is countably compact.

Proof: Let we assume that there exists a compact space (X,τ) which is not countably compact. Hence, $\exists$ $A \subseteq X$, an infinite set having no limit point in X. Thus, each $x \in X$ not necessary a limit point of A. But then for each $x \in X$ there exists an open set G_x containing x such that $G_x \cap [A-\{x\}] = \phi$. Hence either $G_x \cap A = \phi$ or $G_x \cap A = \{x\}, \forall x \in X$. Again $\{G_x\}_{x\in X}$ forms an open cover for X, having restricted sub-cover.

Let $X = \bigcup_{i=1}^{n} G_{x_i}$. Thus, $A = X \cap A = \left[\bigcup_{i=1}^{n} G_{x_i}\right] \cap A = \left[\bigcup_{i=1}^{n} G_{x_i} \cap A\right]$.

As $G_{x_i} \cap A = \{x_i\}$ or $G_{x_i} \cap A = \phi$ $\forall i, 1 \le i \le n$ & $n\left(\bigcup_{i=1}^{n}\left[G_{x_i} \cap A\right]\right) < \infty$.

Hence A is a finite subset of X; a contradiction.
Thus, our assumption is wrong; hence each compact space is countably compact.

Corollary: Every Fort space is a compact space, thereby countably compact and hence Fort space is a countably compact space.

First Axiom Spaces
Let (X,τ) be a topological space. (X,τ) is said to be first axiom if it satisfies the following first axiom of countability.

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For each point $x \in X$, there exists a countable family $\{B_n(x)\}_{n\in N}$ of open sets such that $x \in B_n(x) \subseteq G$ for each $x \in N$ and for any open set G containing x, there exist $n_0 \in N$ such that $x \in B_n(x) \subseteq G$.

The family $\{B_n(x)\}_{n\in N}$ is called a countable local base at *x*.
"If *F* is a closed subset of *X* and *x* is a point of *X* not in *F*, Then there exist a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

Theorem 2.16 A topological space (X,τ) is completely regular if and only if for every $x \in X$ and every open set containing *x*, there exists a continuous mapping $f : X \to [0,1]$ such that $f(x) = 0$ and $f[y] = 1, \forall y \in X - G$.

Proof:- Only if part:
Let *X* be a completely regular $x \in G$ where G is an open set in *X*. Then *X* – *G* is a closed set in *X* with $x \notin X - G$. As *X* is completely regular, $\exists$ a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X - G) = \{1\}$, i.e., $f(y) = 1, \forall y \in X - G$.

If part:
Assume that for every $x \in G$ and every open set containing, there exists a continuous mapping $f : X \to [0,1]$ such that $f(x) = 0$ and $f[y] = 1, \forall y \in X - G$.
To prove that *X* is a completely regular space. Let *F* be a closed set and $x \notin F$.
Then *X* – *F* is an open set containing *x*. Hence by assumption, there exist a continuous real valued function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(X - G) = 1, \forall y \in X - [X - F]$, i.e., $f(x) = 0$ and $f(y) = 1, \forall y \in F$.
Hence (X,τ) is a completely regular space.

Theorem 2.17 Let (X,τ) be completely regular space. Let N be neighbourhood of $x \in X$.
Then there exist a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1, \forall y \in X - N$ and and conversely.

Proof:- As N is neighbourhood of $x \in X \Rightarrow \exists x \in G(open) \subseteq N \,\&\, G \subseteq X$.
Hence $x \notin (X - G)$ where $X - G$ is a closed set in X.
As *X* is completely regular, $\Rightarrow \exists f : X \xrightarrow{\text{continuous}} [0,1]$ such that
$f(x) = 0$ and $f[X - G] = \{1\}, \forall y \in X - N$.
As $x \in G \subseteq N \Rightarrow X - N \subseteq X - G$ we get $f(y) = 1, \forall y \in X - N$.
Conversely, assume that there exist a continuous function $f : X \to [0,1]$ such that

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$f(x)=0$ and $f(y)=1, \forall y \in X-N$.

To prove that (X,τ) be completely regular space. Let G be an open set in X such that $x \in G$.

As G is neighbourhood of $x \in X$ there exist a continuous function $f: X \to [0,1]$ such that $f(x)=0$ and $f(y)=1, \forall y \in X-G$.

Hence by Theorem 2.1, (X,τ) is a completely regular space.

Theorem 2.18 Let (X,τ) be a completely regular space. Let *F* is a closed set in *X* and $x \notin F$. Then $\exists$ a continuous function $f: X \to [0,1]$ such that $f(x)=1$ and $f(F)=\{0\}$.

Proof: Let $x \notin F$ and F be a closed set in X. As X is a completely regular space $\exists$ a continuous function $g: X \to [0,1]$ such that $g(x)=0$ and $g(F)=\{1\}$.

Define the function $g: X \to [0,1]$ by $f(x)=1-g(x), \ \forall x \in X$.

Then *f* is a continuous function and $f(0)=1-g(0)=1-0=1$ and $f(1)=1-g(1)=1-1=0$.

Thus $\exists$ a continuous function $f: X \to [0,1]$ such that $f(0)=1$ and $g(1)=0$.

Theorem 2.19 A homeomorphic image of a completely regular space is a completely regular space.

Proof: - Let (X,τ) be a completely regular space. Let (X^*,τ^*) be any topological space and $f: X \to X^*$ be a homomorphism. To prove that (X^*,τ^*), is a completely regular space. Let F^* be a closed set in (X^*,τ^*), and $x^* \notin F^*$. Since, $f: X \to X^*$ is homorphism, and *f* is onto, then there exist $f(x)=x^*$. As *f* is a continuous function, $f^{-1}[F^*]$ is a closed set in X. Thus, $x^* \notin F^* \Rightarrow x^* \notin f^{-1}[F^*]$

Hence X being a completely regular space, there exist a continuous function $g: X \to [0,1]$ such that $g(x)=0$ and $g\left[f^{-1}([F^*])\right]=\{1\}$. Thus $g\left[f^{-1}(x^*)\right]=0$ and $g\left[f^{-1}([F^*])\right]=\{1\}$

$\Rightarrow \left[g \circ f^{-1}\right](x^*)=0$ and $g\left[f^{-1}([F^*])\right]=\{1\}$.

Now $g \circ f^{-1}: X^* \to [0,1]$ and $f^{-1}: X^* \to X$ is a continuous as *f* is a homeomorphism.

Hence $g \circ f^{-1}$ is a continuous map.

Thus for $x^* \notin F^*$ there exist a continuous function $g \circ f^{-1}: X^* \to [0,1]$ such that

$\Rightarrow \left[g \circ f^{-1}\right](x^*)=0$ and $g\left[f^{-1}([F^*])\right]=\{1\}$.

Hence (X^*,τ^*), is a completely regular space. As homeomorphic image of a completely regular space is a completely regular space, hence (X^*,τ^*) holds each topological property.

Theorem 2.20 Subspace of a completely regular space is a completely regular space.

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Proof:

Let (X,τ) be a completely regular space and suppose (X^*,τ^*) be its subspace. To prove that (X^*,τ^*), is a completely regular space. Let F^* be a closed set in (X^*,τ^*) then $\Rightarrow \exists F(\text{closed}) \subseteq X \big| F^* = F \cap X^*, x^* \notin F^* \Rightarrow x^* \notin F^*(x^* \in X^*)$.

As (X,τ) is a completely regular space and $x^* \notin F^*$, then $\exists f : X \xrightarrow{\text{continuous}} [0,1] \big| f(x^*) = 0 \,\&\, f(F) = \{1\}$.

Let g denote the restriction of f to X^* $\Rightarrow g \in \mathrm{Re}_{f_{\text{continuous}}}$ defined on X^*, such that $g(x^*) = 0$ and $f(F^*) = \{y\}$.

Hence X^*, is a completely regular space. Thus subspace of a completely regular space is a completely regular space. Hence the property of being a completely regular space is a hereditary space.

Theorem 2.21 Every completely regular space is regular.

Proof: Let (X,τ) be a completely regular space. To prove that (X,τ) is regular. Let $x \notin F(closed) \,\&\, x \in X \supseteq F$. As X is a completely regular,

$\exists f : X \xrightarrow{\text{continuous}} [0,1] \big| f(x) = 0 \,\&\, f(F) = \{1\}$.

We know that (R,τ_U) is a Hausdorff space.

Therefore, [0, 1] (being a subspace of (R,τ_U)) is a Hausdorff space.

As $0 \neq 1$ in $[0,1]$, $\exists G,H(\text{open sets}) \subseteq [0,1] \big| G \cap H = \phi; 0 \in G \,\&\, 1 \in H$. But $f : X \to [0,1]$ is continuous $\Rightarrow f^{-1}(G) \in \tau$ and $f^{-1}(H) \in \tau$.

Further $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$. Thus for $x \notin F$ there exist disjoint open sets $f^{-1}(G)$ and $f^{-1}(H)$ in X such that $x \in f^{-1}(G)$ and $F \subseteq f^{-1}(H)$.

Hence (X,τ) is a regular space.

Theorem 2.22 A normal space is completely regular iff it is regular.

Proof: As every completely regular space is a regular space (see Theorem 2.21), the proof of 'only if 'part follows immediately,
To prove if part, assume that X is a normal, regular space. To prove that X is completely regular space. Let F be a closed set and $x \notin F(x \in X)$. Then $X - F$ is an open set containing x.

As X is a regular space, an open set G in X such that $x \in G \subseteq \overline{G} \subseteq X - F$.

As $\overline{G} \subseteq X - F$ we get $\overline{G} \cap F = \phi$

Thus as $\overline{G}$ and F are disjoint closed sets in a normal space X. Hence there exist continuous function that $f : X \to [0,1]$ such that $f(\overline{G}) = \{0\}$ and $f(F) = \{1\}$ (by Urysohn's Lemma).

As $x \in G \subseteq \overline{G} \subseteq X - F$ we get $f(x) = 0$ and $f(F) = \{1\}$. Hence X is a completely regular space.

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Corollary 2.23 Any compact, T_2 – space is completely regular.

Proof:- We know that compact, T_2 – space is both normal and regular (see Theorems …and …T_2 – space) . Hence by Theorem 2.7, any compact, T_2 – space is completely regular.

Corollary 2.24 Any compact, regular space is completely regular.

Proof:- We know that any compact regular space is normal (see Theorem ….Normal spaces)
Hence by Theorem 2.21, it is a completely regular space.

Theorem 2.25 Every locally compact, Hausdorff space is completely regular.

Proof:
Let (X,τ) be a countably compact, Hausdorff space. Let (X^*,τ^*) be unique point compactification of (X,τ) , $X^* = X \cup \{\infty\}$, where $\infty \notin X$ and $\tau^* = \{G \subseteq X^* \mid X^* - G \text{ is a closed compact subset of } X\} \cup \tau$

Claim 1: (X^*,τ^*) is a Hausdorff space.

Let $x \neq y$ in X^*.

Case 1: $x, y \in X$ and $x \neq y$.

As X is a T_2 – space, there exist disjoint open sets G and H in (X,τ) such that $x \in G$ and $y \in H$. But then $G, H \in \tau$, and hence in this case x and y are separated by disjoint open sets in X.

Case 2: $x = \infty, y \in X$ and $x \neq y$.

As X is a locally compact space and $y \in X$ is an interior point of some compact subset say K. Let G be an open set in X such that $x \in G \subseteq K$. As K is a compact subset of a T_2 – space, K is a closed in X and hence $X^* - K$ is open in X^* . Thus $x \in G \subseteq K, \infty \in X - K$ and G, $X^* - K$ are disjoint open sets in X^*.

Thus from both the cases we get (X^*,τ^*) is a Hausdorff space.

Claim 3: (X^*,τ^*) is a completely regular. As (X^*,τ^*) is a compact, Hausdorff space, it is completely regular (By Corollary2.7).

Claim 4: (X,τ) is a completely regular. We know that (X,τ) is a subspace of (X^*,τ^*) and (X^*,τ^*) is a completely regular space. Hence (X,τ) is a completely regular space .

Definition: Completely regular, T_1 – space is called a Tichonov space or a $T_{3\frac{1}{2}}$ space.

Theorem 2.26 Every Tichonov space $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a T_3 – space.

Proof:- As every completely regular space is regular (Theorem 6), every Tichonov space $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a T_3 – space.

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Theorem 2.27 Every space T_4 – space is a Tichonov space $\left(T_{3\frac{1}{2}} \text{ space}\right)$

Proof: Let (X,τ) be a T_4 – space i.e. (X,τ) is a normal T_1 – space. To prove that (X,τ) is a Tichonov space. Let $x \notin F$ where F is a closed set in X $(x \in X)$. As X is a T1 – space, $\{x\}$ is a closed set in X.
$x \notin F \Rightarrow \{x\} \cap F = \phi$. Hence as X is normal there exist a continuous function $f : X \to [0,1]$ such that $f(\{x\}) = \{0\}$ and $f(F) = \{1\}$ (By Urysohn's Lemma)
Thus for $x \notin F$ there exist a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$
Hence X is a completely regular space.

Theorem 2.28 Being Tichonov space $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a topological property.

Proof: - We know that being a completely regular space is a topological property and being a T_1 – space is also a topological property. Hence being a Tichonov $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a topological property.

Theorem 2.29 Being a Tichonov $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a hereditary property.

Proof: We know that being a completely regular space is a hereditary property and being a T_1 – space is a hereditary property. Hence being a Tichonov $\left(T_{3\frac{1}{2}} \text{ space}\right)$ is a hereditary property.

Theorem 2.30 Every Fort space is a completely regular space.

Proof:
Fort's space is a compact, Hausdorff space.
Hence by Corollary2.8, Fort's space is completely regular.
Again as Fort's space is a T_1 – space (being a Hausdorff space) it is a $\left(T_{3\frac{1}{2}} \text{ space}\right)$.

Problem 1. Every metric space is a completely regular space.

Solution: Let (X,d) be a metric space and let τ denote the topology on X induced by the metric *d*. Let F be any closed set in X and $x \notin F$ $(x \in X)$. Then $\{x\} \cap F = \phi$ and $\{x\}$ is a closed set in X. Since (X,τ) is T_1-space. As every metric space is normal (see Normal spaces). By Urysohn's Lemma, there exist a continuous function $f : X \to [0,1]$ such that $f(\{x\}) = \{0\}$ and $f(F) = \{1\}$. But then $f(x) = 0$ and $f(F) = \{1\}$. Therefore (X,τ) is a completely regular space.

After knowing much more about Fort space, its compactness and role of separation axioms in Fort space let us prove some results of topological dynamical systems in Fort space.

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Almost periodic orbits and Minimal sets in Fort's **spaces**
The results of G.D.Birkhof which exhibit the close connection between almost periodic points and minimal sets given by:

Theorem A. Let X be a compact metric space, and $f \in C^0(X)$. Then the closure of every almost periodic orbit of f is a minimal set **[8]**.

Theorem B. Let X be a compact metric space, and $f \in C^0(X)$. Then all points in any minimal set of f are almost periodic points **[8]**.

Theorems A and B are two well-known theorems in topological dynamical systems, which were first raised and proved in 1912 by G.D. Birkhoff and are also discussed in Birkhoff **[1].**

The condition in Theorems A and B that X is a compact metric space can be weakened **[8]**.

Gottschalk [4] proved that Theorem A holds for X being a metric space and Theorem B holds for X being a locally compact metric space.

Gottschalk [5] further proved that Theorem A holds for X being a regular space, and it follows that Theorem B also holds for X being a locally compact regular space.

The following theorem is a generalization of Theorem A.

Theorem 2.31. Let X be Fort space and $f \in C^0(X)$. Then the closure of every almost periodic orbit of f is a minimal set.
Proof: Assume that x is an almost periodic point of the continuous function f, i.e., $f \in C_0(X)$, where X is a Fort space which is an ω-regular space. If closure of $O(x,f)$ is not a minimal set, then $\exists\ y \in \overline{O(x,f)}$ such that x is not the member of orbit generated by function f having seed y, i.e., $x \notin \overline{O(y,f)}$. Since, X is ω-regular, therefore there exist disjoint open sets G_1 and G_2 such that $x \in G_1$ and $O(y,f) \subset G_2$. Since x is almost periodic point of $f \in C^0(X)$, then there exists $n \in \mathrm{N}$ satisfying

$$\left\{ f^{n+k}(x) \middle| k = 0,1,2,\ldots,n \right\} \cap G_1 \neq \phi\ \forall\ n \in \mathrm{N}$$

Let $G = \bigcap_{k=0}^{n} f^{-k}(G_2)$. Then G is an open set containing y. Since $y \in \overline{O(x,f)}$, then there exists positive integer $n \in \mathrm{N}$ such that $f^n \in G$, which infers that $f^{n+k}(x) \in f^k(G) \subset G_2 \subset X - G_1; k = 0,1,2,\ldots,n$

which is a contradiction.

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Thus closure of $O(x,f)$, i.e., $\overline{O(x,f)}$ must be a minimal set. _

This theorem cannot extend to general topological space T_1-spaces, that is, the closure of an almost periodic orbit in a T_1-space may not be a minimal set.

3. MINIMAL SETS IN FORT SPACES

In Theorem 2 of [5], Gottschalk proved that if a locally compact regular space X can be decomposed into the union of minimal sets of some $f \in C^0(X)$, then all points in topological space (X,f) are almost periodic points of f and each minimal set of f is compact. From the first conclusion of Theorem 2 in [5] it is easy to derive the following theorem, which is a generalization of the above Theorem B.

Theorem C. Let X be a locally compact regular space, and $f \in C^0(X)$. Then all points in any minimal set of f are almost periodic points **[8]**.

We now give a generalization of Theorem C to a Fort Space.

Theorem 3.1. Let X be Fort space and $f \in C^0(X)$. Then all points in any minimal set of f are almost periodic points.

Proof. Let M be a minimal set of $f \in C^0(X)$ and $x \in M$. Then x is a recurrent point. Let N_0 be a compact neighbourhood of x. If x is not an almost periodic point of f, then there exist an open neighborhood N of x and positive integers $n_1 < n_2 < n_3 < \ldots$ such that $f^{n_t}(x) \in N \subset N_0$ and $\left\{ f^{n_t+i}(x) \middle| i = 1,2,\ldots,t \right\} \cap N \neq \phi \, \forall t \in N$

Write $y_t = f^{n_t}(x)$ this infers that this relations constitutes an infinite sets consisting of the elements $\{y_1, y_2, y_3, \ldots\}$ in N. Since N_0 is compact, there exists $y \in N_0$ such that any neighbourhood N_1 of y contains infinitely many points in $\{y_1, y_2, y_3, \ldots\}$. Since the minimal set M is closed, we have $y \in M$ and $x \in \overline{O(x,f)} = M$. Thus there exist an $m \in$ N and a neighbourhood N_1 of y such that $f^m(y) \in f^m(N_1) \subset N$. Take $t > m$ such that $y_t \in N$ this infers that $f^{n_t+m}(x) = f^m(y_t) \in f^m(N_1) \subset N$, which is a contradiction. Thus x must be an almost periodic point of f.

This theorem cannot be extended to general topological spaces even to general (nonlocally compact) metric spaces.

Proposition: The map f from the path connected and locally path connected complete metric space X to itself defined is a point-wise recurrent isometric homeomorphism, the closure of any orbit of f is a minimal set, but no point in X is an almost periodic point of f .

Thus, in order to ensure that all minimal sets of any continuous self-map of a space contain only almost periodic points, it is sufficient that the space has the local compactness, and otherwise it may not be sufficient even if the space has many other nice properties such as connectivity and completeness.

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We have pointed out that not all points of minimal sets in non-locally compact metric spaces are almost periodic. However, the following theorem shows that all points of some special minimal sets in arbitrary topological spaces are almost periodic.

Theorem 3.2. Let *X* be Fort space and $f \in C^0(X)$, *and M be a minimal set of f such* that every orbit of *f* in *M* is a finite set. Then all points in *M* are almost periodic points of *f*.

Proof. Suppose *X* be Fort space and $f \in C^0(X)$, Consider any point $x \in M$. Let for any $i \in Z_+$ suppose $x_i = f^i(x)$. Since orbit of any point *x*, i.e., $O(x,f)$ is a finite set, there exist integers $h > k \geq 0$ such that $x_0, x_1, \ldots, x_{n-1}$ are equally different points, and $x_n = x_k$. Let *N* be any neighbourhood of *x*. Since *x* is recurrent, therefore there must exists integers $m \geq k$ such that $x_m \in N$, which infers that $x_{m+i(n-k)} = x_m \in N \forall i \in N$. Thus *x* is an almost periodic point of *f*.
Thus, it can be conclude that:

Corollary. Let *X* be Fort space and $f \in C^0(X)$, and *M* be a minimal set of *f* such that every orbit of *f* in *M* is a finite set. Then all points in *M* are almost periodic points of *f*.

Let us now see the application of compactness of minimal sets.
Theorem D. Let *X* be a locally compact regular space, and $f \in C^0(X)$. Then each minimal set of *f* is compact.

Theorem 3.8. Let *X* be a Fort space which is either Hausdorff or regular and $f \in C^0(X)$.
Then each minimal set of *f* is compact.

Proof. Suppose *X* be Fort space and $f \in C^0(X)$. Fort space is Hausdorff space as well as regular space. Let *M* be a minimal set of *f* and $x \in M$, and let *N* be a compact neighborhood of *x*. Then by the definition of Fort space and compactness, *x* is almost periodic point of *X*. Thus there exists neighbourhood N_x of *x* and positive integer *t* such that $f^{n_t}(x) \in N_x$ and $\left\{ f^{n+k}(x) \middle| k = 1,2,\ldots,t \right\} \cap N_x \neq \phi \forall t \in \mathbb{N}$

Let $W = \bigcup_{k=0}^{n} f^k(N_x)$, then *W* is the collection of functional of neighbourhoods, as $f \in C^0(X)$ implying *f* preserve the mapping, i.e., image of open sets is again open set. Thus, *W* is compact. This infers that orbit of every point of *f* is the member of *W, i.e.*, $O(x,f) \subset W$. Since for any point $\xi \in X$ has a compact and closed neighborhood N_y . Consider some finite sub-covers $\left\{ N_{y_1}, N_{y_2}, \ldots, N_{y_m} \right\}$ from the cover $\left\{ N_y : y \in W \right\}$ of *W* and suppose $Z = \bigcup_{k=1}^{m} N_{y_k}$. As *Z* is the union of sub-covers, hence *Z* is compact set closed set and hence $O(x,f) \subset W \subset Z$ this infers that $V = \overline{O(x,f)} \subset Z$
Hence, as a closed subset of the compact set *Z* , *W* is compact.

The above theorem is the extension of Theorem C to general locally compact topological spaces.

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4. POINTS IN CLOSURES OF ALMOST PERIODIC ORBITS

Using Theorems A and B we obtained the following theorem **[8]**.

Theorem E. Let *X* be a compact metric space, and $f \in C^0(X)$.Then all points in the closure of any almost periodic orbit of *f* are almost periodic.

Following theorem is the generalized form of Theorem E given by:

Theorem F. Let *X* be a locally compact *ω-regular space,* and $f \in C^0(X)$. Then all points in the closure of any almost periodic orbit of *f* are almost periodic.

Theorem 4.1. Let *X* be Fort space and $f \in C^0(X)$. Then all points in the closure of any almost periodic orbit of *f* are almost periodic.

Proof. Let *x* be an almost periodic point of $f \in C^0(X)$, where *X* is a Fort space which is an *ω-regular space*, let $y \in \overline{O(x,f)}$, and let N_y be an open neighbourhood of *y*. Then $\exists\ t \in \mathbb{N}$ and an open neighbourhood N_x of *x* such that $f^t(N_x) \subset N_y$.

Claim 1. *There exists* $t \in \mathbb{N}$ *such that*

$$\{f^{n+k}(x) | k = 1,2,\ldots,t\} \cap N_x \neq \phi \,\forall\, t \in \mathbb{N}$$

In fact, if Claim 1 is not true, then there exist integers $0 < n_1 < n_2 < n_3 < \ldots$ such that

$$\{f^{n_\lambda+k}(x) | k = 1,2,\ldots,t\} \cap N_x = \phi \,\forall\, t \in \mathbb{N}.$$

Let $A = \bigcup_{\lambda=1}^{\infty} \{f^{n_\lambda+k}(x) | k = 0,1,2,\ldots,t\} \Rightarrow A$ is a countable set in $X - N_x$. Since *X* is *ω*-regular and $X - N_x$ is closed, there exist disjoint open sets G_1 and G_2 in *X* such that $x \in G_1 \subset N_x$ and $A \subset G_2$. Since *x* is an almost periodic point, there exists an $\exists\ t \in \mathbb{N}$ such that

$$\{f^{\mu+k}(x) | k = 0,1,2,\ldots,m\} \cap G_1 \neq \phi \quad \forall\, \mu \in Z_+$$

Let $y_t = f^{n_t}(y)$. Then $\{y_t, f(y_t), f^2(y_t), \ldots, f^m(y_t)\} \subset Y \subset G_1$ Let $G_2 = \bigcap_{k=0}^{m} f^{-k}(G_1)$. Then V_2 is an open set containing y_t. Since $y_t \in O(y,f) \subset \overline{O(x,f)}$, there is a $\mu \in \mathbb{N}$ such that $f^{\mu}(x) \in V_2$, this infers that $\{f^{\mu+k}(x) | k = 0,1,2,\ldots,m\} \subset \bigcup_{k=0}^{m} f^k(V_2) \subset V_1$ and hence, we have

$\{f^{\mu+k}(x) | k = 0,1,2,\ldots,m\} \cap G_1 = \phi$, which is a contradiction. Thus Claim 1 must be true.

It follows from Claim 1 that $\{f^{\mu+k}(x) | k = 0,1,2,\ldots,m\} \cap G_1 \neq \phi \ \forall\, \mu \in Z_+$. Hence, *y* is an almost periodic point of *f*.

Remark. The condition in Theorem 4.1 that *X* is an *ω*-regular space cannot be replaced by that *X* is a T_1-space.

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